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PERIODIC SOLUTIONS OF LINEAR SYSTEMS WITH A LAG

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PERIODIC SOLUTIONS OF LINEAR SYSTEMS WITH A LAG

by A. Halanay

The necessary and adequate conditions for the existence of periodic solutions to linear inhomogeneous systems with a lag are established in this work. In view of the fact that the particular case of differential equation systems with a lagging argument is a matter of special interest, we will begin by outlining such a case.

1. Under consideration is the system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), \tag{1}$$

where A, B, f, are continuous and periodic functions of a $\omega > \tau$ period.

We will also consider the homogeneous system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) \tag{2}$$

and the corresponding conjugate system

$$\dot{y}(t) = -y(t)A(t) - y(t+\tau)B(t+\tau) \tag{3}$$

We will prove by direct calculation that if the system (1) has periodic solutions of the ω period, then $\int_0^\omega y(t)f(t)\,dt = 0$ for all periodic solutions of y(t), period ω, system (3).

Let x(t) be the solution of system (1), determined with $t \geqslant -\tau$, and y(t) the solution of system (3), determined with $t \leqslant \omega + \tau$.

With $0 \leqslant t \leqslant \omega$, we shall designate

$$(y, x) = y(t)x(t) + \int_0^\tau y(t+\eta)B(t+\eta)x(t+\eta-\tau)\,d\eta = y(t)x(t) + \\ + \int_t^{t+\tau} y(\xi)B(\xi)x(\xi-\tau)\,d\xi.$$

We have $\frac{d}{dt}(y,x) = y(t)f(t)$; hence, if x(t) and y(t) are periodic functions of the ω period, we will get

$$\int_0^\omega y(t)f(t)\,dt = \int_0^\omega \frac{d}{dt}(y,x)\,dt = (y,x)_\omega - (y,x)_0 = 0$$

and our assertion has been proved.

2. Let $Y(\alpha, t)$ be a matrix whose lines, being a function of α, satisfy system (3) with $\alpha < t$ and conditions $Y(t, t) = E$, $Y(\alpha, t) \equiv 0$ when $t < \alpha \leqslant t + \tau$ where E is a unit matrix. Let $x(t)$ be the solution of system (1), determined with $t \geqslant \sigma - \tau$. We have

$$\int_\sigma^t Y(\alpha, t)\dot{x}(\alpha)\,d\alpha = \int_\sigma^t Y(\alpha, t)A(\alpha)x(\alpha)\,d\alpha + \int_\sigma^t Y(\alpha, t)B(\alpha)x(\alpha-\tau)\,d\alpha + \int_\sigma^t Y(\alpha, t)f(\alpha)\,d\alpha.$$

Further,

$$Y(t, t)x(t) - Y(\sigma, t)x(\sigma) - \int_\sigma^t \left[\frac{\partial}{\partial\alpha} Y(\alpha, t)\right] x(\alpha)\,d\alpha = \int_\sigma^t Y(\alpha, t)A(\alpha)x(\alpha)\,d\alpha + \int_{\sigma-\tau}^{t-\tau} Y(\beta+\tau, t)B(\beta+\tau)x(\beta)\,d\beta + \int_\sigma^t Y(\alpha, t)f(\alpha)\,d\alpha.$$

Hence we get

$$x(t) = Y(\sigma, t)x(\sigma) - \int_{t-\tau}^t Y(\alpha+\tau, t)B(\alpha+\tau)x(\alpha)\,d\alpha + \int_{\sigma-\tau}^\sigma Y(\alpha+\tau, t)B(\alpha+\tau)x(\alpha)\,d\alpha + \int_\sigma^t Y(\alpha, t)f(\alpha)\,d\alpha.$$

But when $t - \tau < \alpha \leqslant t$ we have $t < \alpha + \tau \leqslant t + \tau$, consequently $Y(\alpha+\tau, t) \equiv 0$.

Finally

$$x(t) = Y(\sigma, t)x(\sigma) + \int_{\sigma-\tau}^\sigma Y(\alpha+\tau, t)B(\alpha+\tau)x(\alpha)\,d\alpha + \int_\sigma^t Y(\alpha, t)f(\alpha)\,d\alpha. \qquad (4)$$

With $f \equiv 0$, this formula produces a general solution of system (2). Let us examine matrix $X(t, \sigma)$ whose columns are solutions of system (2), which are equal to zero when $\sigma - \tau \leqslant t < \sigma$, and such that $X(\sigma, \sigma) = E$. It follows from formula (4) that $X(t, \sigma) = Y(\sigma, t)$.

We will now establish a corresponding formula for system (3). Let $y(t)$ -- the solution of system (3), defined with $t \leqslant \sigma + \tau$, $X(\alpha, t)$, -- be the above-constructed matrix $(\alpha \geqslant t - \tau)$.

We have

$$\int_t^\sigma \dot{y}(s) X(s, t)\, ds = -\int_t^\sigma y(s) A(s) X(s, t)\, ds - \int_t^\sigma y(s+\tau) B(s+\tau) X(s, t)\, ds.$$

Hence

$$y(\sigma) X(\sigma, t) - y(t) X(t, t) - \int_t^\sigma y(s) \frac{\partial}{\partial s} X(s, t)\, ds =$$
$$= -\int_t^\sigma y(s) A(s) X(s, t)\, ds - \int_{t+\tau}^{\sigma+\tau} y(\beta) B(\beta) X(\beta - \tau, t)\, d\beta.$$

Further,

$$y(t) = y(\sigma) X(\sigma, t) - \int_t^\sigma y(s) A(s) X(s, t)\, ds - \int_t^\sigma y(s) B(s) X(s-\tau, t)\, ds +$$
$$+ \int_t^\sigma y(s) A(s) X(s, t)\, ds + \int_{t+\tau}^{t} y(s) B(s) X(s-\tau, t)\, ds +$$
$$+ \int_t^\sigma y(s) B(s) X(s-\tau, t)\, ds + \int_t^{\sigma+\tau} y(s) B(s) X(s-\tau, t)\, ds.$$

Since with $t \leqslant s < t+\tau$ we have $t - \tau \leqslant s - \tau < t$, then $X(s-\tau, t) \equiv 0$,
and consequently,

$$y(t) = y(\sigma) X(\sigma, t) + \int_\sigma^{\sigma+\tau} y(s) B(s) X(s-\tau, t)\, ds$$

or

$$y(t) = y(\sigma) Y(t, \sigma) + \int_\sigma^{\sigma+\tau} y(s) B(s) Y(t, s-\tau)\, ds. \qquad (5)$$

3. Let $x(t, \varphi)$ be the solution of system (1), defined with $t \geqslant -\tau$ which at $[-\tau, 0]$ coincides with the preset continuous function φ. It follows from the fact that A, B, f are periodic functions of the ω period, that $x(t+\omega, \varphi)$ is also a solution of the system determined with $t+\omega \geqslant -\tau$, and, consequently, with $t \geqslant -\omega - \tau$. If, when $\tau \leqslant t \leqslant 0$, that solution coincides with φ, then $x(t+\omega, \varphi) \equiv x(t, \varphi)$ on the basis of the uniqueness theorem. Therefore, the condition required to make the solution periodic looks like the following $x(\omega + s, \varphi) = \varphi(s)$ with $s \in [-\tau, 0]$. Let V be the operator defined by the formula $V\varphi = x(\omega + s; \varphi)$; φ is the initial function for a periodic solution only when $V\varphi = \varphi$. Let $z(t; \varphi)$ be the solution of system (2) which at $[-\tau, 0]$ coincides with φ. It follows from formula

(4) that

$$x(t, \varphi) = z(t; \varphi) + \int_{\sigma}^{t} Y(\alpha, t) f(\alpha)\, d\alpha.$$

If U is the operator defined by formula $U\varphi = z(\omega + s; \varphi)$, then

$$V\varphi = U\varphi + \int_{0}^{\omega + s} Y(\alpha, \omega + s) f(\alpha)\, d\alpha.$$

The condition $V\varphi = \varphi$ appears as

$$\varphi = U\varphi + \int_{0}^{\omega + s} Y(\alpha, \omega + s) f(\alpha)\, d\alpha$$

or as

$$(I - U)\varphi = \int_{0}^{\omega + s} Y(\alpha, \omega + s) f(\alpha)\, d\alpha,$$

where I is an identical operator.

It can easily be verified that if $\omega > \tau$, then the operator U is fully continuous, if the space of functions φ is a Banach [Russian term: banakhovoye] space of continuous vector-functions preset at $[-\tau, 0]$ with the norm $\|\varphi\| = \sup_{-\tau \leqslant s \leqslant 0} |\varphi(s)|$. Actually, this is what we get from (4)

$$U\varphi = Y(0, \omega + s)\varphi(0) + \int_{-\tau}^{0} Y(\alpha + \tau, \omega + s) B(\alpha + \tau) \varphi(\alpha)\, d\alpha \qquad (6)$$

and it is apparent that if $\|\varphi\| < M$, then $\|U\varphi\| < L(M)$, and it follows from $\dot{z}(\omega + s; \varphi) = A(s) z(\omega + s; \varphi) + B(s) z(\omega + s - \tau; \varphi)$ that the derivatives of the functions $U\varphi$ are uniformly limited; consequently, if $\{\varphi\}$ is a limited set, $\{U\varphi\}$ is a compact set; in this case, essential use was made of the fact that $\omega > \tau$.

Thus system (1) has a periodic solution, with all f, only if I — U has an inverse operator; as U is a fully continuous operator I, U has an inverse operator only if the equation $U\varphi = \varphi$ has a null solution. But the solutions of this equation are the initial functions for the periodic solutions of system (2); consequently, system (1) has a periodic solution of period ω, with all f, only if system (2) has no other periodic solution of period ω, except a trivial solution. If system (2) has a periodic solution of period ω, then $U\varphi = \varphi$ can have no null solutions. It follows from the general theory of fully continuous operators that this equation has only a finite number of independent periodic solutions of period ω.

A similar reasoning may be applied also to system (3). Let $y(t;\psi)$ be the solution of system (3), defined with $t \leqslant \omega + \tau$, which coincides with ψ at $[\omega, \omega+\tau]$. Together with $y(t;\psi)$, $y(t-\omega;\psi)$ is also a solution. If $y(t-\omega;\psi)$ coincides with ψ at $[\omega, \omega+\tau]$, then $y(t-\omega;\psi) \equiv y(t;\psi)$. Consequently, ψ will be the initial function for the periodic solution of system (3) only if $y(t-\omega;\psi) = \psi(t)$, with $t \in [\omega, \omega+\tau]$. According to formula (5), this condition looks like the following

$$\psi(t) = \psi(\omega)\, Y(t-\omega, \omega) + \int_{\omega}^{\omega+\tau} \psi(\xi)\, B(\xi)\, Y(t-\omega, \xi-\tau)\, d\xi. \tag{7}$$

We will designate $\tilde{\varphi}(s) = \psi(s+\omega+\tau),\ -\tau \leqslant s \leqslant 0$. Then, if ψ is the solution of equation (7), $\tilde{\varphi}$ is the solution of the following equation

$$\tilde{\varphi}(s) = \tilde{\varphi}(-\tau)\, Y(s+\tau, \omega) + \int_{\omega}^{\omega+\tau} \tilde{\varphi}(\xi-\omega-\tau)\, B(\xi)\, Y(s+\tau, \xi-\tau)\, d\xi$$

or

$$\tilde{\varphi}(s) = \tilde{\varphi}(-\tau)\, Y(s+\tau, \omega) + \int_{-\tau}^{0} \tilde{\varphi}(\eta)\, B(\eta+\tau)\, Y(s+\tau, \eta+\omega)\, d\eta. \tag{8}$$

4. We will show that if the equation $\varphi - U\varphi = 0$, where U is given with formula (6), and equation (8) have the same number of linearly independent solutions, and that the equation $\varphi - U\varphi = F(s)$ has a solution only if

$$\tilde{\varphi}(-\tau)F(0) + \int_{-\tau}^{0} \tilde{\varphi}(\eta)\, B(\eta+\tau)\, F(\eta)\, d\eta = 0$$

applies to all solutions $\tilde{\varphi}(s)$ of equation (8).

Let us examine the following equation

$$\varphi(s) - K_1(s, -\tau)\,\varphi(0) - \int_{-\tau}^{0} K_1(s, \eta)\, B(\eta+\tau)\, \varphi(\eta)\, d\eta = \chi(s). \tag{9}$$

If $|K_1|$ is small enough, this equation will have the following solution $\varphi(s) = \sum_{i=0}^{\infty} \varphi_i(s)$, where $\varphi_i(s)$ is recurrently defined as

$$\varphi_0(s) = \chi(s),\ \varphi_i(s) = K_1(s, -\tau)\,\varphi_{i-1}(0) + \int_{-\tau}^{0} K_1(s, \eta) B(\eta + \tau)\,\varphi_{i-1}(\eta) d\eta.$$

We get

$$\varphi_i(s) = K_i(s, -\tau)\,\chi(0) + \int_{-\tau}^{0} K_i(s, \eta) B(\eta + \tau)\,\chi(\eta)\, d\eta,$$

where

$$K_i(s, \eta) = K_1(s, -\tau) K_{i-1}(0, \eta) + \int_{-\tau}^{0} K_1(s, \zeta)\, B(\zeta + \tau)\, K_{i-1}(\zeta, \eta)\, d\zeta.$$

By designating $\Gamma(s, \eta) = \sum_{1}^{\infty} K_i(s, \eta)$, the solution of equation (9) will look like the following

$$\varphi(s) = \chi(s) + \Gamma(s, -\tau)\chi(0) + \int_{-\tau}^{0} \Gamma(s, \eta) B(\eta + \tau)\chi(\eta) d\eta. \tag{10}$$

We will now examine equation

$$\tilde{\varphi}(s) - \tilde{\varphi}(-\tau) K_1(0, s) - \int_{-\tau}^{0} \tilde{\varphi}(\eta) B(\eta + \tau) K_1(\eta, s)\, d\eta = \tilde{\chi}(s). \tag{11}$$

If $|K_1|$ is small enough, this equation will have the following solution $\tilde{\varphi}(s) = \sum_{i=1}^{\infty} \tilde{\varphi}_i(s)$, where

$$\tilde{\varphi}_0(s) = \tilde{\chi}(s),\ \tilde{\varphi}_i(s) = \tilde{\varphi}_{i-1}(-\tau) K_1(0, s) + \int_{-\tau}^{0} \tilde{\varphi}_{i-1}(\eta) B(\eta + \tau)\, K_1(\eta, s)\, d\eta.$$

We get

$$\tilde{\varphi}_i(s) = \tilde{\chi}(-\tau) \tilde{K}_i(0, s) + \int_{-\tau}^{0} \tilde{\chi}(\eta) B(\eta + \tau) \tilde{K}_i(\eta, s) d\eta,$$

where

$$\tilde{K}_i(\eta, s) = \tilde{K}_{i-1}(\eta, -\tau) K_1(0, s) + \int_{-\tau}^{0} \tilde{K}_{i-1}(\eta, \zeta) B(\zeta + \tau) K_1(\zeta, s) d\zeta.$$

If $\tilde{\Gamma}(\eta, s) = \sum_{i=1}^{\infty} \tilde{K}_i(\eta, s)$, the solution of equation (11) will look like the following

$$\tilde{\varphi}(s) = \tilde{\chi}(s) + \tilde{\chi}(-\tau)\,\tilde{\Gamma}(0, s) + \int_{-\tau}^{0} \tilde{\chi}(\eta) B(\eta + \tau)\,\tilde{\Gamma}(\eta, s) d\eta. \tag{12}$$

It is easy to show by the induction method that $\tilde{K}_i(\eta, s) = K_i(\eta, s)$, and consequently $\widetilde{\Gamma}(\eta, s) = \Gamma(\eta, s)$, and (12) appears as follows

$$\widetilde{\varphi}(s) = \widetilde{\chi}(s) + \widetilde{\chi}(-\tau)\,\Gamma(0, s) + \int_{-\tau}^{0} \widetilde{\chi}(\eta)\, B(\eta + \tau)\,\Gamma(\eta, s)\, d\eta. \tag{13}$$

It follows from the fact that $Y(\eta + \tau, \omega + s)$ is continuous, when $-\tau \leqslant \eta \leqslant 0$, $-\tau \leqslant s \leqslant 0$, that

$$Y(\eta + \tau, \omega + s) = \sum_k a_k(s)\, b_k(\eta) + K_1(s, \eta), \tag{14}$$

where $a_k(s)$ are column vectors, $b_k(\eta)$ line vectors, a_k and b_k are linearly independent and $|K_1|$ may be as small as possible.

The equation $\varphi - U\varphi = F(s)$ will then look like this

$$\varphi(s) - K_1(s, -\tau)\,\varphi(0) - \int_{-\tau}^{0} K_1(s, \eta)\, B(\eta + \tau)\,\varphi(\eta)\, d\eta =$$
$$= \sum_k a_k(s)\, b_k(-\tau)\,\varphi(0) + \sum_k \int_{-\tau}^{0} a_k(s)\, b_k(\eta)\, B(\eta + \tau)\,\varphi(\eta)\, d\eta + F(s).$$

We will designate

$$\varphi(s) - K_1(s, -\tau)\,\varphi(0) - \int_{-\tau}^{0} K_1(s, \eta)\, B(\eta + \tau)\,\varphi(\eta)\, d\eta = \chi(s).$$

Then, if formula (10) is used

$$\chi(s) = \sum_k a_k(s)\, b_k(-\tau) \Big[\chi(0) - \Gamma(0, -\tau)\,\chi(0) +$$
$$+ \int_{-\tau}^{0} \Gamma(0, \eta)\, B(\eta + \tau)\,\chi(\eta)\, d\eta \Big] + \sum_k \int_{-\tau}^{0} a_k(s)\, b_k(\eta)\, B(\eta + \tau) \Big[\chi(\eta) +$$
$$+ \Gamma(\eta, -\tau)\,\chi(0) + \int_{-\tau}^{0} \Gamma(\eta, \zeta)\, B(\zeta + \tau)\,\chi(\zeta)\, d\zeta \Big] d\eta + F(s).$$

We will designate

$$\bar{b}_k(\eta) = b_k(\eta) + b_k(-\tau)\,\Gamma(0, \eta) + \int_{-\tau}^{0} b_k(\zeta)\, B(\zeta + \tau)\,\Gamma(\zeta, \eta)\, d\zeta.$$

Then the equation will appear like this

$$\chi(s) = \sum_k a_k(s)\, \bar{b}_k(-\tau)\,\chi(0) + \sum_k a_k(s) \int_{-\tau} b_k(\eta)\, B(\eta + \tau)\,\chi(\eta)\, d\eta + F(s).$$

Hence, $\chi(s) - F(s) = \sum_k \lambda_k a_k(s)$. For λ_k we get the following system

$$\lambda_k = \sum_j \gamma_{kj}\, \lambda_j + f_k. \tag{15}$$

where

$$\gamma_{kj} = \bar{b}_k(-\tau)\,a_j(0) + \int_{-\tau}^{0} b_k(\eta)\,B(\eta+\tau)\,a_j(\eta)\,d\eta,$$

$$f_k = \bar{b}_k(-\tau)\,F(0) + \int_{-\tau}^{0} \bar{b}_k(\eta)\,B(\eta+\tau)\,F(\eta)\,d\eta.$$

System (15) has a solution only if

$$\sum f_k \mu_k = 0 \tag{16}$$

applies to all the solutions of the following system

$$\mu_k = \sum_j \gamma_{jk}\,\mu_j. \tag{17}$$

Condition (16) is necessary and adequate to the solution of $\varphi - U\varphi = F(s)$.

Taking (14) into account, the equation (8) will look like this

$$\tilde{\varphi}(s) - \tilde{\varphi}(-\tau)\,K_1(0,s) - \int_{-\tau}^{0} \tilde{\varphi}(\eta)\,B(\eta+\tau)\,K_1(\eta,s)\,d\eta =$$
$$= \sum_k \tilde{\varphi}(-\tau)\,a_k(0)\,b_k(s) + \sum_k \int_{-\tau}^{0} \tilde{\varphi}(\eta)\,B(\eta+\tau)\,a_k(\eta)\,b_k(s)\,d\eta.$$

If we were to designate

$$\tilde{\varphi}(s) - \tilde{\varphi}(-\tau)\,K_1(0,s) - \int_{-\tau}^{0} \tilde{\varphi}(\eta)\,B(\eta+\tau)\,K_1(\eta,s)\,d\eta = \tilde{\chi}(s)$$

and use (13), the result would be

$$\tilde{\chi}(s) = \Big[\tilde{\chi}(-\tau) + \tilde{\chi}(-\tau)\,\Gamma(0,-\tau) +$$
$$+ \int_{-\tau}^{0} \tilde{\chi}(\eta)\,B(\eta+\tau)\,\Gamma(\eta,-\tau)\,d\eta\Big] \sum_k a_k(0)\,b_k(s) + \sum_k \int_{-\tau}^{0} \Big[\tilde{\chi}(\eta) +$$
$$+ \tilde{\chi}(-\tau)\,\Gamma(0,\eta) + \int_{-\tau}^{0} \tilde{\chi}(\zeta)\,B(\zeta+\tau)\,\Gamma(\zeta,\eta)\,d\zeta\Big] B(\eta+\tau)\,a_k(\eta)\,b_k(s)\,d\eta.$$

Let

$$\tilde{a}_k(\eta) = F(\eta,-\tau)\,a_k(0) + a_k(\eta) + \int_{-\tau}^{0} \Gamma(\eta,\zeta)\,B(\zeta+\tau)\,a_k(\zeta)\,d\zeta.$$

Then

$$\tilde{\chi}(s) = \tilde{\chi}(-\tau) \sum_k \tilde{a}_k(0)\,b_k(s) + \sum_k \int_{-\tau}^{0} \tilde{\chi}(\eta)\,B(\eta+\tau)\,\tilde{a}_k(\eta)\,b_k(s)\,d\eta.$$

The solution of this equation will be $\tilde{\chi}(s) = \sum_k \mu_k b_k(s)$, where

$$\mu_k = \sum_j \tilde{\gamma}_{kj} \mu_j, \quad \tilde{\gamma}_{kj} = b_j(-\tau)\,\tilde{a}_k(0) + \int_{-\tau}^{0} b_j(\eta)\, B(\eta + \tau)\, \tilde{a}_k(\eta)\, d\eta. \tag{18}$$

It can be verified by direct calculation that $\tilde{\gamma}_{kj} = \gamma_{jk}$; consequently, system (18) coincides with (17). Condition (16) appears as follows

$$\sum_k \mu_k \bar{b}_k(-\tau) F(0) + \sum_k \int_{-\tau}^{0} \mu_k b_k(\eta)\, B(\eta + \tau)\, F(\eta)\, d\eta = 0.$$

After some simple calculations, bearing in mind that

$$\sum \mu_k b_k(s) = \tilde{\chi}(s),$$

we get

$$\left[\tilde{\chi}(-\tau) + \tilde{\chi}(-\tau)\,\Gamma(0, -\tau) + \int_{-\tau}^{0} \tilde{\chi}(\zeta)\, B(\zeta + \tau)\, \Gamma(\zeta, -\tau)\, d\zeta\right] F(0) +$$
$$+ \int_{-\tau}^{0} \Big[\tilde{\chi}(\eta) + \tilde{\chi}(-\tau)\,\Gamma(0, \eta) +$$
$$+ \int_{-\tau}^{0} \tilde{\chi}(\zeta)\, B(\zeta + \tau)\, \Gamma(\zeta, \eta)\, d\zeta\Big]\, B(\eta + \tau) F(\eta)\, d\eta = 0,$$

that is

$$\tilde{\varphi}(-\tau) F(0) + \int_{-\tau}^{0} \tilde{\varphi}(\eta)\, B(\eta + \tau)\, F(\eta)\, d\eta = 0.$$

5. We have $F(s) = \int_0^{\omega + s} Y(\alpha, \omega + s) f(\alpha)\, d\alpha$ and $\tilde{\varphi}(s) = \psi(s + \tau + \omega)$. We then get the following succession

$$\tilde{\varphi}(-\tau) F(0) + \int_{-\tau}^{0} \tilde{\varphi}(\eta)\, B(\eta + \tau)\, F(\eta)\, d\eta = \psi(\omega) \int_0^{\omega} Y(\alpha, \omega) f(\alpha)\, d\alpha +$$
$$+ \int_{-\tau}^{0} \psi(\eta + \tau + \omega)\, B(\eta + \tau) \left[\int_0^{\omega + \eta} Y(\alpha, \omega + \eta) f(\alpha)\, d\alpha\right] d\eta =$$
$$= \psi(\omega) \int_0^{\omega} Y(\alpha, \omega) f(\alpha)\, d\alpha +$$
$$+ \int_0^{\omega - \tau} \left[\int_{-\tau}^{0} \psi(\eta + \tau + \omega)\, B(\eta + \tau)\, Y(\alpha, \omega + \eta)\, d\eta\right] f(\alpha)\, d\alpha +$$
$$+ \int_{\omega - \tau}^{\omega} \left[\int_{\alpha - \omega}^{0} \psi(\eta + \tau + \omega)\, B(\eta + \tau)\, Y(\alpha, \omega + \eta)\, d\eta\right] f(\alpha)\, d\alpha =$$
$$= \psi(\omega) \int_0^{\omega} Y(\alpha, \omega) f(\alpha)\, d\alpha +$$
$$+ \int_0^{\omega} \left[\int_{-\tau}^{0} \psi(\eta + \tau + \omega)\, B(\eta + \tau)\, Y(\alpha, \omega + \eta)\, d\eta\right] f(\alpha)\, d\alpha =$$
$$= \int_0^{\omega} \left[\psi(\omega)\, Y(\alpha, \omega) + \int_{\omega}^{\omega + \tau} \psi(\xi)\, B(\xi)\, Y(\alpha, \xi - \tau)\, d\xi\right] f(\alpha)\, d\alpha =$$
$$= \int_0^{\omega} y(\alpha) f(\alpha)\, d\alpha,$$

where y(α) is the solution of system (3) defined by the initial function ψ. Since $\psi(s+\tau+\omega)=\tilde{\varphi}(s)$ and $\tilde{\varphi}$ is the solution of equation (8), y(α) is the periodic solution of system (3). This produces

THEOREM 1. The necessary and adequate condition required in order that system (1) may have a periodic solution of period ω is the fulfillment of equality $\int_0^\omega y_r(\alpha)f(\alpha)d\alpha=0$. for all linearly independent periodic solutions of period ω for system (3).
[System (3) has the same number of linearly-independent periodic solutions of period ω as system (2)].

6. Let us examine a general system with a lag

$$\dot{x}(t)=\int_{-\infty}^{0}x(t+s)d_s\eta(t,s)+f(t), \tag{19}$$

where

a) η(t, s) is defined with t $\geqslant$ 0 as $-\infty<s<+\infty,\ \eta(t,s)\equiv 0$ with s $\geqslant$ 0;

b) there exist functions $\tau_{ij}(t)$ and $V_{ij}(t)$, which are limited when t $\geqslant$ 0, and such that $\eta_{ij}(t,s)\equiv\eta_{ij}(t,-\tau_{ij}(t))\equiv 0$ when $s\leqslant-\tau_{ij}(t),\ \underset{s=-\tau_{ij}(t)}{\overset{0}{V}}\eta_{ij}(t,s)\leqslant V_{ij}(t)$ where, as usual, $\underset{s=\alpha}{\overset{\beta}{V}}f$ means a complete change of function f by [α, β];

c) η_{ij}(t, s) are continuous with respect to t, and uniform in relation to s;

d) η(t, s), f(t), $\tau_{ij}(t)$, $V_{ij}(t)$ are periodic by t, period ω.

Eventually, the number indicated by τ will be such that $\tau\geqslant\tau_{ij}(t)$; then $\eta(t,s)\equiv 0$ with $s\leqslant-\tau$. Let Y(α,t) be a matrix satisfying the system

$$Y(\alpha,t)+\int_{-\infty}^{0}\eta(\alpha-\gamma,\gamma)Y(\alpha-\gamma,t)d\gamma=\text{const}$$

and conditions $Y(\alpha,t)\equiv 0$, with $\alpha>t,\ Y(t,t)=E$. Then $Y(\alpha-\gamma,t)\equiv 0$ with $\gamma<\alpha-t$, and we get

$$Y(\alpha,t)+\int_{\alpha-t}^{0}\eta(\alpha-\gamma,\gamma)Y(\alpha-\gamma,t)d\gamma=\text{const}$$

or

$$Y(\alpha, t) + \int_\alpha^t \eta(\beta, \alpha - \beta) Y(\beta, t)\, d\beta = E.$$

It is independently verified that the solution of this equation, constructed by successive approximations, is continuous by (α, t) with a limited change by α.

Let x(t) be the solution of system (19). Then

$$\int_\sigma^t \dot{x}(\alpha) Y(\alpha, t)\, d\alpha = \int_\sigma^t \left[\int_{-\infty}^0 x(\alpha + s)\, d_s \eta(\alpha, s)\right] Y(\alpha, t)\, d\alpha + \\ + \int_\sigma^t f(\alpha) Y(\alpha, t)\, d\alpha.$$

Hence

$$x(t) Y(t, t) - x(\sigma) Y(\sigma, t) - \int_\sigma^t x(\alpha)\, d_\alpha Y(\alpha, t) =$$
$$= \int_\sigma^t \left[\int_{-\infty}^\alpha x(s)\, d_s \eta(\alpha, s - \alpha)\right] Y(\alpha, t)\, d\alpha + \int_\sigma^t f(\alpha)\, Y(\alpha, t)\, d\alpha =$$
$$= \int_{-\infty}^\sigma x(s)\, d_s \int_\sigma^t \eta(\alpha, s - \alpha) Y(\alpha, t)\, d\alpha +$$
$$+ \int_\sigma^t x(s)\, d_s \int_s^t \eta(\alpha, s - \alpha) Y(\alpha, t)\, d\alpha + \int_\sigma^t f(\alpha) Y(\alpha, t)\, d\alpha.$$

Taking into consideration the conditions for η(t,s) and Y(α, t), we get

$$x(t) = x(\sigma) Y(\sigma, t) + \int_{\sigma-\tau}^\sigma x(s)\, d_s \int_\sigma^{\sigma+\tau} \eta(\alpha, s - \alpha) Y(\alpha, t)\, d\alpha + \\ + \int_\sigma^t f(\alpha) Y(\alpha, t)\, d\alpha. \tag{20}$$

If X(t, σ) is a matrix whose lines, given t $> \sigma$, satisfy system (18) with f $\equiv$ 0 and conditions $X(\sigma, \sigma) = E,\ X(t, \sigma) \equiv 0$, given t $< \sigma$, then it follows from (20) that $X(t, \sigma) = Y(\sigma, t)$.

We will therefore examine the conjugate system

$$y(\alpha) + \int_{-\infty}^0 \eta(\alpha - \gamma, \gamma) y(\alpha - \gamma)\, d\gamma = \text{const.} \tag{21}$$

As $\eta(t, s) \equiv 0$, when s $\geqslant$ 0 and $s \leqslant -\tau$, the system (21) looks as follows

$$y(\alpha) + \int_\alpha^{\alpha+\tau} \eta(\beta, \alpha - \beta) y(\beta)\, d\beta = \text{const},$$

consequently, for a fixed σ

$$y(\alpha) + \int_\alpha^\sigma \eta(\beta, \alpha - \beta) y(\beta) d\beta = y(\sigma) + \int_\sigma^{\sigma+\tau} \eta(\beta, \sigma - \beta) y(\beta) d\beta - \\ - \int_\sigma^{\alpha+\tau} \eta(\beta, \alpha - \beta) y(\beta) d\beta.$$

It may be seen from this that if the solution of y(t) is sought on the $[\sigma, \sigma+\tau]$, intercept, it is defined, given $t < \sigma$, by a system of integral equations of the Volterra type; this makes it possible also to formulate a theorem of the existence of uniqueness for this system. The solution to the initial function with a limited change by $[\sigma, \sigma+\tau]$ is found in the class of functions with a limited change; if the initial function is continuous, the solution is also continuous.

Let $t < \sigma, X(\alpha, \gamma)$ be a matrix whose lines satisfy both the functions of system (19) with $f \equiv 0$ and conditions $X(\alpha, \gamma) \equiv 0$ with $\alpha < \gamma$, $X(\gamma, \gamma) = E$. We have to solve $y(\alpha)$ of system (21).

$$\int_t^\sigma X(\alpha, t) dy(\alpha) = X(\sigma, t) y(\sigma) - X(t, t) y(t) - \int_t^\sigma \left[\frac{\partial}{\partial \alpha} X(\alpha, t)\right] y_i'(\alpha) d\alpha.$$

Hence

$$y(t) = X(\sigma, t) y(\sigma) + \int_t^\sigma X(\alpha, t) d_\alpha \left[\int_{-\infty}^0 \eta(\alpha - \gamma, \gamma) y(\alpha - \gamma) d\gamma\right] - \\ - \int_t^\sigma \left[\int_{-\infty}^0 X(\alpha + s, t) d_s \eta(\alpha, s)\right] y(\alpha) d\alpha.$$

After changing the order of integration in the last integral, we get

$$y(t) = X(\sigma, t) y(\sigma) + \int_{\sigma-\tau}^\sigma X(\beta, t) d_\beta \int_\sigma^{\sigma+\tau} \eta(\alpha, \beta - \alpha) y(\alpha) d\alpha. \tag{22}$$

7. (21) shows that if $y(\alpha)$ is the solution of the equation, then $y(\alpha - \omega)$ will also be a solution, as $\eta(t, s)$ is periodic by t, period ω. Let $y(\alpha, \psi)$ be a solution defined at $\alpha < \sigma$ with a definition of function ψ by $[\sigma, \sigma+\tau]$; such a solution is provided by formula (22). Function $y(\alpha - \omega, \psi)$ will also be a solution, and this solution is defined when $\alpha - \omega < \sigma$, that is when $\alpha < \omega + \sigma$. If, when $\sigma \leqslant \alpha \leqslant \sigma + \tau$, this solution coincides with Ψ, then it will coincide with $y(\alpha, \psi)$ when $\alpha < \sigma$,

and consequently the solution $y(\alpha, \psi)$ will be periodic of period ω. The periodicity condition of the $y(\alpha, \psi)$ solution is therefore expressed as $y(\alpha-\omega, \psi)=\psi(\alpha)$ when $\sigma \leqslant \alpha \leqslant \sigma+\tau$. We have

$$y(\alpha-\omega, \psi)=X(\sigma, \alpha-\omega)\psi(\sigma)+$$
$$+\int_{\sigma-\tau}^{\sigma} X(\beta, \alpha-\omega)\, d_\beta \int_{\sigma}^{\sigma+\tau} \eta(\gamma, \beta-\gamma)\psi(\gamma)\, d\gamma.$$

We should point out also that $X(\sigma, t) = Y(t, \sigma)$. Then the initial functions of the periodic solution of system (21) will be the solutions of the following equation

$$\psi(\alpha)=Y(\alpha-\omega, \sigma)\psi(\sigma)+\int_{\sigma-\tau}^{\sigma} Y(\alpha-\omega, \beta)\, d_\beta \int_{\sigma}^{\sigma+\tau} \eta(\gamma, \beta-\gamma)\psi(\gamma)\, d\gamma.$$

Let $\sigma=\omega$, $\tilde{\varphi}(s)=\psi(s+\omega+\tau)$; we get

$$\tilde{\varphi}(s)=Y(s+\tau, \omega)\tilde{\varphi}(-\tau)+\int_{-\tau}^{0} Y(s+\tau, \beta+\omega)\, d_\beta \int_{-\tau}^{0} \eta(\gamma+\tau, \beta-\gamma-\tau)\tilde{\varphi}(\gamma)\, d\gamma. \qquad (23)$$

The same reasoning and the use of formula (2) show that the initial functions of the periodic solutions of system (19) are the solutions of the following equation

$$\varphi(s)-\varphi(0)Y(0, \omega+s)-\int_{-\tau}^{0}\varphi(\beta)\, d_\beta \int_{-\tau}^{0} \eta(\alpha+\tau, \beta-\alpha-\tau)Y(\alpha+\tau, \omega+s)\, d\alpha=$$
$$=\int_{0}^{\omega+s} f(\alpha)Y(\alpha, \omega+s)\, d\alpha. \qquad (24)$$

8. Let us examine the following equation

$$\varphi(s)-\varphi(0)K_1(-\tau, s)-\int_{-\tau}^{0}\varphi(\beta)\, d_\beta \int_{-\tau}^{0} \eta(\alpha+\tau, \beta-\alpha-\tau)K_1(\alpha, s)\, d\alpha=\chi(s). \qquad (25)$$

If $|K_1|$ is small enough, the solution of this equation will appear as $\varphi(s)=\sum_{i=0}^{\infty}\varphi_i(s)$, where $\varphi_0(s)=\chi(s)$,

$$\varphi_i(s)=\varphi_{i-1}(0)K_1(-\tau, s)+\int_{-\tau}^{0}\varphi_{i-1}(\beta)\, d_\beta \int_{-\tau}^{0} \eta(\alpha+\tau, \beta-\alpha-\tau)K_1(\alpha, s)\, d\alpha.$$

Further, we have

$$\varphi_i(s)=\chi(0)K_i(-\tau, s)+\int_{-\tau}^{0}\chi(\beta)\, d_\beta \int_{-\tau}^{0} \eta(\alpha+\tau, \beta-\alpha-\tau)K_i(\alpha, s)\, d\alpha,$$

where

$$K_i(\alpha, s)=K_{i-1}(\alpha, 0)K_1(-\tau, s)+$$
$$+\int_{-\tau}^{0} K_{i-1}(\alpha, \gamma)\, d_\gamma \int_{-\tau}^{0} \eta(\xi+\tau, \gamma-\xi-\tau)K_1(\xi, s)\, d\xi.$$

If $\Gamma(\alpha, s) = \sum_{1}^{\infty} K_i(\alpha, s)$, we get the following solution for equation (25)

$$\varphi(s) = \chi(s) + \chi(0)\,\Gamma(-\tau, s) + \int_{-\tau}^{0} \chi(\beta)\, d_\beta \int_{-\tau}^{0} \eta(\alpha+\tau, \beta-\alpha-\tau)\,\Gamma(\alpha, s)\, d\alpha. \tag{26}$$

We will now examine this equation

$$\tilde{\varphi}(s) - K_1(s, 0)\,\tilde{\varphi}(-\tau) - \int_{-\tau}^{0} K_1(s, \beta)\, d_\beta \int_{-\tau}^{0} \eta(\gamma+\tau, \beta-\gamma-\tau)\,\tilde{\varphi}(\gamma)\, d\gamma = \tilde{\chi}(s). \tag{27}$$

If $|K_1|$ is small enough, the solution of the equation will appear as follows $\tilde{\varphi}(s) = \sum_{i=0}^{\infty} \tilde{\varphi}_i(s)$, where $\tilde{\varphi}_0(s) = \tilde{\chi}(s)$,

$$\tilde{\varphi}_i(s) = K_1(s, 0)\,\tilde{\varphi}_{i-1}(-\tau) + \int_{-\tau}^{0} K_1(s, \beta)\, d_\beta \int_{-\tau}^{0} \eta(\gamma+\tau, \beta-\gamma-\tau)\,\tilde{\varphi}_{i-1}(\gamma)\, d\gamma.$$

Further, we have

$$\tilde{\varphi}_i(s) = \tilde{K}_i(s, 0)\,\tilde{\chi}(-\tau) + \int_{-\tau}^{0} \tilde{K}_i(s, \beta)\, d_\beta \int_{-\tau}^{0} \eta(\gamma+\tau, \beta-\gamma-\tau)\,\tilde{\chi}(\gamma)\, d\gamma,$$

where

$$\tilde{K}_i(s, \beta) = K_1(s, 0)\,\tilde{K}_{i-1}(-\tau, \beta) + \\ + \int_{-\tau}^{0} K_1(s, \alpha)\, d_\alpha \int_{-\tau}^{0} \eta(\gamma+\tau, \alpha-\gamma-\tau)\,\tilde{K}_{i-1}(\gamma, \beta)\, d\gamma.$$

The induction method is used to prove that $\tilde{K}_i(s, \beta) = K_i(s, \beta)$. The solution of equation (27) thus appears as follows

$$\tilde{\varphi}(s) = \tilde{\chi}(s) + \Gamma(s, 0)\,\tilde{\chi}(-\tau) + \int_{-\tau}^{0} \Gamma(s, \beta)\, d_\beta \int_{-\tau}^{0} \eta(\gamma+\tau, \beta-\gamma-\tau)\,\tilde{\chi}(\gamma)\, d\gamma. \tag{28}$$

9. As $Y(\alpha+\tau, \omega+s)$ is equi-continuous when $-\tau \leqslant \alpha \leqslant 0$, $\tau \leqslant s \leqslant 0$, it is possible to write down

$$Y(\alpha+\tau, \omega+s) = \sum_k a_k(\alpha)\, b_k(s) + K_1(\alpha, s) \tag{29}$$

where $a_k(\alpha)$ are vector columns, $b_k(s)$ vector lines, a_k, b_k linearly independent and $|K_1|$ may be as small as possible. Taking (29) into consideration, the equation (24) will look like

$$\varphi(s) - \varphi(0)K_1(-\tau, s) - \int_{-\tau}^{0}\varphi(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)K_1(\alpha, s)\,d\alpha =$$
$$= \varphi(0)\sum_k a_k(-\tau)b_k(s) +$$
$$+\sum_k\int_{-\tau}^{0}\varphi(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)a_k(\alpha)b_k(s)\,d\alpha + F(s),$$

where

$$F(s) = \int_0^{\omega}f(\alpha)\,Y(\alpha, \omega+s)\,d\alpha.$$

Let

$$\varphi(s) - \varphi(0)K_1(-\tau, s) - \int_{-\tau}^{0}\varphi(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)K_1(\alpha, s)\,d\alpha = \chi(s).$$

Taking (26) into consideration, we get the following equation for $\chi(s)$:

$$\chi(s) = \Big[\chi(0) + \chi(0)\Gamma(-\tau, 0) +$$
$$+\int_{-\tau}^{0}\chi(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)\Gamma(\alpha, 0)\,d\alpha\Big]\sum_k a_k(-\tau)b_k(s) +$$
$$+\sum_k\int_{-\tau}^{0}\Big[\chi(\beta) + \chi(0)\Gamma(-\tau, \beta) + \int_{-\tau}^{0}\chi(\xi)\,d\xi\int_{-\tau}^{0}\eta(\alpha+\tau, \xi-$$
$$-\alpha-\tau)\Gamma(\alpha, \beta)\,d\alpha\Big]d\beta\int_{-\tau}^{0}\eta(\sigma+\tau, \beta-\sigma+\tau)a_k(\sigma)b_k(s)\,d\sigma + F(s).$$

If we designate

$$\bar{a}_k(\alpha) = \Gamma(\alpha, 0)a_k(-\tau) + a_k(\alpha) + \int_{-\tau}^{0}\Gamma(\alpha, \xi)\,d\xi\int_{-\tau}^{0}\eta(\sigma+\tau, \xi-\sigma-\tau)a_k(\sigma)\,d\sigma,$$

then we obtain

$$\chi(s) = \chi(0)\sum_k \bar{a}_k(-\tau)b_k(s) + \sum_k\int_{-\tau}^{0}\chi(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-$$
$$-\alpha-\tau)\bar{a}_k(\alpha)b_k(s)\,d\alpha + F(s).$$

It is apparent from the foregoing that $\chi(s) - F(s) = \sum_k \lambda_k b_k(s)$, in which $\lambda_k = \sum_j \gamma_{kj}\lambda_j + f_k$,

$$\gamma_{kj} = b_j(0)\bar{a}_k(-\tau) + \int_{-\tau}^{0}b_j(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)\bar{a}_k(\alpha)\,d\alpha,$$
$$f_k = F(0)\bar{a}_k(-\tau) + \int_{-\tau}^{0}F(\beta)\,d\beta\int_{-\tau}^{0}\eta(\alpha+\tau, \beta-\alpha-\tau)\bar{a}_k(\alpha)\,d\alpha. \tag{30}$$

System (30) has a solution, only if

$$\sum_k \mu_k f_k = 0 \tag{31}$$

applies to all solutions of the system

$$\mu_k = \sum_j \gamma_{jk} \mu_j \tag{32}$$

As (30) is equivalent to the equation for $\chi(s)$, and this equation is equivalent to equation (24), we conclude that condition (31) represents the necessary and adequate condition for the existence of periodic solutions of period ω to system (19).

On the basis of (29), equation (23) is expressed as

$$\tilde{\varphi}(s) - K_1(s, 0)\tilde{\varphi}(-\tau) - \int_{-\tau}^{0} K_1(s, \beta)\, d\beta \int_{-\tau}^{0} \eta(\gamma + \tau, \beta - \gamma - \tau)\tilde{\varphi}(\gamma)\, d\gamma =$$
$$= \sum_k a_k(s) b_k(0)\tilde{\varphi}(-\tau) + \sum_k \int_{-\tau}^{0} a_k(s) b_k(\beta)\, d\beta \int_{-\tau}^{0} \eta(\gamma +$$
$$+ \tau, \beta - \gamma - \tau)\tilde{\varphi}(\gamma)\, d\gamma.$$

Designating

$$\tilde{\varphi}(s) - K_1(s, 0)\tilde{\varphi}(-\tau) - \int_{-\tau}^{0} K_1(s, \beta)\, d\beta \int_{-\tau}^{0} \eta(\gamma + \tau, \beta - \gamma - \tau)\tilde{\varphi}(\gamma)\, d\gamma = \tilde{\chi}(s)$$

and using (28), we get

$$\tilde{\chi}(s) = \sum_k a_k(s)\bar{b}_k(0)\tilde{\chi}(-\tau) + \sum_k a_k(s)\int_{-\tau}^{0} b_k(\beta)\, d\beta \int_{-\tau}^{0} \eta(\gamma + \tau, \beta -$$
$$- \gamma - \tau)\tilde{\chi}(\gamma)\, d\gamma,$$

where

$$\bar{b}_k(\beta) = b_k(\beta) + b_k(0)\Gamma(-\tau, \beta) + \int_{-\tau}^{0} b_k(\sigma)\, d\sigma \int_{-\tau}^{0} \eta(\gamma + \tau, \sigma - \gamma - \tau)\Gamma(\gamma, \beta)\, d\gamma.$$

Thus

$$\tilde{\chi}(s) = \sum_k \mu_k a_k(s), \quad \text{где } \mu_i = \sum_j \tilde{\gamma}_{ki}\mu_j$$

$$\tilde{\gamma}_{kj} = \bar{b}_k(0) a_j(-\tau) + \int_{-\tau}^{0} b_k(\beta)\, d\beta \int_{-\tau}^{0} \eta(\gamma + \tau, \beta - \gamma - \tau) a_j(\gamma)\, d\gamma.$$

It is then verified by direct calculation that $\tilde{\gamma}_{kj} = \gamma_{jk}$. Then, if μ_k is the solution to system (32), we get

[Translator's note: Some of Russian copy provided for this page is not legible.]

$$\sum_k \mu_k f_k = F(0)[\Gamma(-\tau, 0)\tilde{\chi}(-\tau) + \tilde{\chi}(-\tau) +$$
$$+ \int_{-\tau}^{0} \Gamma(-\tau, \xi)\, d\xi \int_{-\tau}^{0} \eta(\sigma + \tau, \xi - \sigma - \tau)\tilde{\chi}(\sigma)\, d\sigma] +$$
$$+ \int_{-\tau}^{0} F(\beta)\, d\beta \int_{-\tau}^{0} \eta(\alpha + \tau, \beta - \alpha - \tau)\Big[\tilde{\chi}(\alpha) + \Gamma(\alpha, 0)\tilde{\chi}(-\tau) +$$
$$+ \int_{-\tau}^{0} \Gamma(\alpha, \xi)\, d\xi \int_{-\tau}^{0} \eta(\sigma + \tau, \xi - \sigma - \tau)\tilde{\chi}(\sigma)\, d\sigma\Big]\, d\alpha =$$
$$= F(0)\tilde{\varphi}(-\tau) + \int_{-\tau}^{0} F(\beta)\, d\beta \int_{-\tau}^{0} \eta(\alpha + \tau, \beta - \alpha - \tau)\tilde{\varphi}(\alpha)\, d\alpha$$

10. It follows from the preceding calculations that the necessary and adequate condition for the existence of periodic solutions of period to system (19) may be written down as

$$\int_0^\omega f(\alpha)\, Y(\alpha,\omega)\, d\alpha\, \tilde{\varphi}(-\tau) + \int_{-\tau}^{0} \left(\int_0^{\omega+\beta} f(\alpha)\, Y(\alpha, \omega+\beta)\, d\alpha \right) d_\beta \int_{-\tau}^{0} \eta(\alpha + \tau, \beta - \alpha - \tau)\, \tilde{\varphi}(\alpha)\, d\alpha = 0$$

applicable to all solutions $\tilde{\varphi}$ of equations (23). But $\tilde{\varphi}(\xi) = \psi(\alpha + \omega + \tau)$, and we get the following sequence

$$\int_0^\omega f(\alpha) Y(\alpha,\omega)\, d\alpha\, \tilde{\varphi}(-\tau) + \int_{-\tau}^{0} \left(\int_0^{\omega+\beta} f(\alpha)\, Y(\alpha, \omega+\beta)\, d\alpha \right) d_\beta \int_{-\tau}^{0} \eta(\alpha + \tau, \beta - \alpha - \tau)\, \tilde{\varphi}(\alpha)\, d\alpha = \int_0^\omega f(\alpha)\, Y(\alpha,\omega)\, \psi(\omega)\, d\alpha +$$

$$+ \int_{-\tau}^{0} \left(\int_0^{\omega+\beta} f(\alpha)\, Y(\alpha, \omega+\beta)\, d\alpha \right) d_\beta \int_\omega^{\omega-\tau} \eta(\xi, \beta - \xi + \omega)\, \psi(\xi)\, d\xi =$$

$$= \int_0^\omega f(\alpha)\, Y(\alpha,\omega)\, \psi(\omega)\, d\alpha +$$

$$+ \int_{-\omega}^{0} \left(\int_0^{\omega+\beta} f(\alpha)\, Y(\alpha, \omega+\beta)\, d\alpha \right) d_\beta \int_\omega^{\omega+\tau} \eta(\xi, \beta - \xi + \omega)\, \psi(\xi)\, d\xi =$$

$$= \int_0^\omega f(\alpha) \left[Y(\alpha,\omega)\, \psi(\omega) + \int_\alpha^\omega Y(\alpha,\gamma)\, d_\gamma \int_\omega^{\omega+\tau} \eta(\xi, \gamma - \xi)\, \psi(\xi)\, d\xi \right] d\alpha =$$

$$= \int_0^\omega f(\alpha)\, y(\alpha)\, d\alpha,$$

where $y(\alpha)$ is the solution of system (21), defined by the initial function of ψ. Thus we get

THEOREM 2. The necessary and adequate condition required in order that system (19) may have a periodic solution of period ω is the fulfillment of the following equality

$$\int_0^\omega f(\alpha)\, y_k(\alpha)\, d\alpha = 0$$

for all linearly-independent periodic solution of period ω of system (21). [With f $\equiv$ 0, system (21) and system (19) have the same number of linearly-independent periodic solution of period ω].